Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(99)00020-9

J. Appl. Maths Mechs, Vol. 63, No. 1, pp. 141–144, 1999

0021-8928/99/$—see front matter

THE PROBLEM OF A WEDGE-SHAPED PUNCH ON THE FACE OF AN ELASTIC WEDGE†

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(*Received* 13 *March* 1997)

The behaviour of the contact stresses at a singular point, which is the intersection of the vertex of a wedge-shaped punch, with an apex angle that is not too small, with the edge of an elastic three-dimensional wedge, is investigated by a numerical-analytical method [1] for different types of boundary conditions on its other face. Agreement between the numerical results and the results obtained previously by an asymptotic method for small apex angles [2] with respect to real singularity exponents is established. The reason why the numerical-analytical method [1] is unable to obtain the well-known singularity with exponent $-3/2$ per $+ i\omega$[2, 3] is investigated. The conclusion drawn in [2] on the occurrence of terms of the order of $r^{-3/2-\omega_1-i\omega_2}$, $0 < \omega_1 < 1/2$ for small angles of the elastic wedge, one face of which is rigidly clamped, is confirmed. This leads to stronger oscillations of the contact pressure in the neighbourhood of the vertex of the punch. The mechanism by which new oscillating terms arise for an elastic wedge with one stress-face is indicated. The mutual effect of two similar wedge-shaped punches on a half-space is investigated as a special case. © 1999 Elsevier Science Ltd. All rights reserved.

Suppose a punch, wedge-shaped in planform, is pressed into one face of an elastic three-dimensional wedge with apex angle α ($0 < \alpha < 2\pi$), so that the contact area Ω is an infinite wedge (see Fig. 1) with apex angle 2β ($0 < \beta < \pi/2$), described in polar coordinates ρ, ψ ($r = \rho\cos\psi$, $z = \rho\sin\psi$, the z axis coincides with the edge of the elastic wedge) by the inequalities $0 \leqslant \rho < \infty$, $|\psi| < \beta$. The base of the punch is described by the function $f(\rho, \psi)$, $(\rho, \psi) \in \Omega$. One of the following boundary conditions is satisfied on the other face of the elastic wedge: (a) there are no stresses, (b) sliding contact, and (c) rigid bonding. The normal contact stress function $q(\rho, \psi)$, $(\rho, \psi) \in \Omega$, referred to $\theta = G/(1-\nu)$, is unknown, where G is the shear modulus and ν is Poisson's ratio.

To eliminate the solutions of the contact problem with infinite energy we will henceforth assume that a Mellin transformation with respect to the variable ρ can be applied to the functions $\mathbf{q}(\rho, \psi)$ and $f(\rho, \psi)$, and

$$\int_{-\beta}^{\beta} d\psi \int_0^\infty q(\rho,\psi)\rho d\rho < \infty, \quad \int_{-\beta}^{\beta} d\psi \int_0^\infty f(\rho,\psi)\rho d\rho < \infty \tag{1}$$

The integral equation of contact problems a, b and c, after applying the integral Mellin transformation is obtained in the form [2]

$$\int_{-1}^{1} q_s(\xi)K_s(\beta\xi,\beta x)d\xi = f_s(x), \quad |x| \leqslant 1 \tag{2}$$

$$K_s = K_s(t,p) = \frac{1}{2\cos\pi s}P_{s-1/2}(-\cos(t-p)) + K_s^*(t,p) \tag{3}$$

$$K_s^*(t,p) = \frac{1}{2\pi}\int_0^\infty \operatorname{sh}\pi u(W_j(u) - \operatorname{cth}\pi u)[R_+(-s,u,t)R_+(s,u,p) + R_-(-s,u,t)R_-(s,u,p)]du +$$

$$+\frac{1}{\pi}\int_0^\infty \operatorname{sh}\frac{\pi u}{2} W_j(u)\left[R_+(s,u,p)B_j^u\left\{\operatorname{ch}\frac{\pi y}{2}R_+(-s,y,t)\right\}+\right.$$

$$\left.+R_-(s,u,p)B_j^u\left\{\operatorname{ch}\frac{\pi y}{2}R_-(-s,y,t)\right\}\right]du \quad (|\operatorname{Re} s| < 1/2) \tag{4}$$

where we have introduced new functions and independent variables by the formulae

$$x = \psi/\beta, \quad q_s(x) = q_s^*(\psi), \quad f_s(x) = f_s^*(\psi)/\beta$$

$$\frac{1}{2\pi i}\int_\Gamma q_s^*(\psi)\rho^{-s-3/2}ds = q(\rho,\psi), \quad \frac{1}{2\pi i}\int_\Gamma f_s^*(\psi)\rho^{-s-1/2}ds = f(\rho,\psi) \tag{5}$$

†*Prikl. Mat. Mekh.* Vol. 63, No. 1, pp. 152–156, 1999.

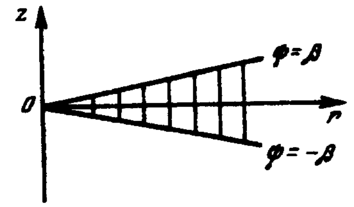

Fig. 1.

and we have also used the notation employed in (1.3) and (1.6) in [2]. The following misprints in formulae (1.3) in [2] need to be corrected: in the denominator $W_1(u)$, there should be a plus sign instead of the last sign, both pairs must be selected in the expression for $W_\pm(u)$, in the formula for $g_2(t)$ the denominator should have the form ch $2\alpha t - \cos 4\alpha$, in the first row of the expression for $g_3(t)$ there should be $tf_3(t)$ instead of $ft_3(t)$, while in the third row there should be $\sin\alpha$ instead of $\sin\theta$.

In (4) the values $m = 1, 2, 3$ correspond to problems a, b, c, respectively. In (5) Γ is the straight line in the plane of the complex variable s, parallel to the imaginary axis which intersects the real axis slightly to the left of the point $s = 1/2$.

As was proved in [1], the exponent of the singularity of the contact-pressure function $q(\rho, \psi)$ when $\rho \to 0$ is related to the points of the spectrum of the integral operator on the left-hand side of Eq. (2). The poles s_k of the function $q_s(\xi)$ will obviously be those values of the parameter s for which non-trivial solutions of the corresponding homogeneous equation can exist, i.e. points of the spectrum of the integral operator (2). Here s_k is independent of ξ.

To find s_k we carry out a discretization of Eq. (2) using the Bubnov–Galerkin method: the solution is sought in the form of an expansion in a system of basis functions $\upsilon_m(\xi)$

$$q_s(\xi) = \sum_{m=0}^{\infty} t_m(s) v_m(\xi) \tag{6}$$

while to determine $t_m = t_m(s)$ the residual is decomposed with respect to the second basis $\{u_l\}_{l=0}^{\infty}$; as a result we obtain the following infinite system with respect to the unknowns

$$\sum_{m=0}^{\infty} a_{lm} t_m = f_l, \quad l = 0, 1, \ldots$$

$$a_{lm} = (K_s v_m, w_l)|_{L_2} = \int_{-1}^{1}\int_{-1}^{1} K_s(\beta\xi, \beta x) v_m(\xi) w_l(x)\, d\xi\, dx$$

$$f_l = \int_{-1}^{1} f_s(x) w_l(x)\, dx \tag{7}$$

Here $\{w_k\}_{k=0}^{\infty}$ is a system of projectors onto the basis $\{u_l\}_{l=0}^{\infty}$, i.e. $(u_l, w_k)|_{L_2} = \delta_{kl}$, where δ_{kl} is the Kronecker delta.

The function $q_s(\xi)$ when $\xi = \pm 1$ has a weak singularity of the form $(1 - \xi^2)^{-1/2}$. To regularize the initial ill-posed problem—integral equation (2)—we need to take this singularity into account in the coordinate functions. Hence, it is natural to select as the basis functions the system of functions [1]

$$v_m(\xi) = \frac{T_m(\xi)}{\sqrt{1 - \xi^2}}; \quad u_l(x) = c_l T_l(x), \quad c_0 = 1, \quad c_l = 2 \quad (l \geq 1) \tag{8}$$

where $T_m(x)$ are Chebyshev polynomials of the first kind. No singularity occurs in the second system, since the right-hand side and the residual are smooth functions. In view of the condition for the Chebyshev polynomials to be orthogonal we obtain that

$$w_k(x) = \frac{T_k(x)}{\pi\sqrt{1 - x^2}} \tag{9}$$

Using the expansion of a Legendre function of the form [1]

$$\frac{1}{2\cos\pi s} P_{s-1/2}(-\cos(t - p)) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} G_k(s) e^{ik(t-p)}$$

$$G_k(s)=\frac{\Gamma(0.5s+0.5|k|+0.25)\Gamma(-0.59s+0.5|k|+0.25)}{2\Gamma(0.5s+0.5|k|+0.75)\Gamma(-0.5s+0.5|k|+0.75)} \tag{10}$$

and the value of the integral [4]

$$\int_{-1}^{1} e^{\pm iax}\frac{T_m(x)}{\sqrt{1-x^2}}dx=\pi e^{\pm im\pi/2}J_m(a) \tag{11}$$

we obtain the following expression for the elements of the matrix of system (7) $a_{lm} = a_{lm}(s)$ $(l, m = 0, 1, \ldots)$

$$a_{lm}(s)=\sum_{n=0}^{\infty}{}' G_n(s)J_m(\beta n)J_l(\beta n)\cos\left(\frac{l-m}{2}\pi\right)+\frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1} K_s^*(\beta x,\beta y)\frac{T_m(x)T_l(y)}{\sqrt{1-x^2}\sqrt{1-y^2}}dx\,dy \tag{12}$$

The prime on the summation sign denotes that the first term ($n = 0$) of the series in (12) is taken with the coefficient 1/2. If, when $s = s_k$, the determinant $D(s)$ of the infinite-dimensional matrix with elements (12) vanishes, then $q(\rho, \psi) \sim \rho^{-\gamma}$, $\gamma = 3/2 + s_k$ when $\rho \to 0$. When $\beta > 0.1\pi$ and for values of α that are not too small (for which calculations will be carried out below, it is sufficient to reduce the matrix with elements (12) to dimension 4–5, so as to ensure three correct significant figures for the zeros of its determinant on the real axis [1].

Note that when $s \in \mathbf{R}$ the functions $R_\pm(s, u, t)$ (3) take real values; to calculate the values of the Legendre functions, occurring in the expression for $R_\pm(s, u, t)$, it is convenient to use their representation in terms of the hypergeometric function [4].

Instead of summing the Neumann series $B^u_{\pm 2,3}$, which occur in formula (1.3) in [2], we solve the Fredholm integral equations of the second kind numerically, the analytical solutions of which are represented by these series. These Fredholm equations can be found in [5]. To solve them we used the method of mechanical quadratures with the Gauss quadrature formula for 32 nodes [6].

In Table 1 we give the values of the largest singularity exponent γ for problems a, b and c, corresponding to real zeros $D(s)$, $s \in (-3/2; -1/2)$ as a function of the angles α and β (in degrees); all the calculations are carried out for $\nu = 0.3$.

For problem a with $\alpha = 225°$, 270° and 315° and the same values of 2β, given in Table 1, the values of γ differ by less than 6% from the corresponding values given in the last column of the table, which (for a wedge on a half-space) agree well with the curve shown in Fig. 4.2 in [1] for $\varepsilon = 0$.

By the theorem of implicit functions $s_k(\alpha, \beta)$—the root of the equation $D(s) = 0$, depends analytically on α and β. Consequently, as α and β change continuously these roots trace out continuous curves in the s plane.

As calculations show, for problem a with $2\beta = \pi$ and $\alpha = 100°$ in the range $\in (-3/2; -1/2)$ close to the point $s = -1/2$ there are two additional zeros of $D(s)$ ($\gamma_1 = 0.98$ and $\gamma_2 = 0.96$), which, if we fix α and reduce the angle 2β, merge into a quadratic root giving a singularity of the order of $\rho^{-\gamma}(C_1 + C_2 \ln \rho)$, and then converge to the real axis and become complex conjugates, leading to oscillations of the contact-pressure function as $\rho \to 0$ and separation of the tip of the punch from the elastic base.

For problem c, for sufficiently acute angles α, zeros of $D(s)$ occur at $s \in (0; 1/2)$; for example, when $\alpha = 0.1\pi$, $2\beta = 45°$, $s_1 = 0.45$, $s_2 = 0.47$. On combining and emerging into the complex domain, these zeros lead to the strongest singularity of order $\gamma = \omega_1 + i\omega_2 + 3/2$, $\omega \in (0; 1/2)$ which confirms the results obtained in [2] by an asymptotic method.

Problem b is obviously equivalent to the problem of the symmetrical pressing of two similar wedge-shaped punches of different faces of a wedge of twice the apex angle. In particular, comparison of the last column of the table for

Table 1

Problem	2β	α						
		45	90	135	180	225	270	315
	45	0.999	0.985	0.862	0.788	–	–	–
a	90	0.999	0.987	0.740	0.703	–	–	–
	135	0.999	0.990	0.632	0.611	–	–	–
	45	0.448	0.528	0.447	0.390	0.411	0.422	0.412
b	90	0.224	0.295	0.224	0.180	0.194	0.204	0.195
	135	0.158	0.080	0.246	0.416	0.353	0.310	0.350
	45	0.626	0.685	0.704	0.711	0.714	0.719	0.721
c	90	0.555	0.602	0.622	0.629	0.632	0.637	0.640
	135	0.520	0.542	0.555	0.559	0.561	0.565	0.567

problem a (one punch on a half-space) and the column $\alpha = 90°$ for problem b (two punches on a half-space) enables the degree of mutual influence of the two wedge-shaped punches on the half-plane to be estimated: weaker singularities arise here than for a single punch.

Note that in the case of two wedge-shaped punches with a plane base on a half-space, an accurate solution of the contact problem was obtained by Rvachev [7, p. 206]. However, Rvachev's solution does not contain the singularities obtained as above as $\rho \to 0$. The reason for this deficiency of the solution is the fact that the deformation energy (the integral over the contact area of the product of the contact-pressure functions and the normal displacement) becomes infinite. These solutions are eliminated here since a Mellin transformation with respect to ρ cannot be applied to the function $f(\rho, \psi)$ = const (a punch with a flat base), and conditions (1) also break down.

Although the numerical method described above is not very effective when $\beta \to 0$, its results for real roots are close to those obtained in [2] by the asymptotic method. For values of β that are not too small, the existence of a root in the range $s \in (-3/2; -1/2)$ of the real axis is ensured by the fact that when $s = -3/2$ and $s = -1/2$ the function $D(s)$ has multiple poles, since they are single for each element (12) like the poles of the gamma functions that occur in the function $G_n(s)$ (10); when $s \in (-3/2; -1/2)$, $D(s)$ has different signs in the region of these poles (this can be verified numerically).

On the other hand, in the case of an elastic half-space, as here for a wedge of arbitrary apex angle, a pure imaginary root of the equation $D(s) = 0$ was obtained in [1] using the Bubnov–Galerkin method, leading to the singularity $\gamma = 3/2 + i\theta^*$, found for a half-space [3] and for a wedge [2]. This can obviously be explained as follows. Since the value of θ_* is usually fairly large [2], while the elements of the matrix (12) decrease strongly as $\operatorname{Im} s \to \infty$, to calculate the determinant $D(s)$ one must increase the order of the reduced matrix and, in fact, find the limit of the sum of an infinitely large number of terms of infinitely small quantities, for which the numerical-analytic method [1] is quite unsuccessful. Nevertheless, it can be asserted that the singularity $\gamma = 3/2 + i\theta^*$ in the case, for example, of an elastic half-space, as follows from formulae (1.13) in [3], is present at least up to the value $\lambda = 2$ (the angle of the punch is 57.3°), when, up to terms $O(\lambda^{-6}\ln^3\lambda), \theta^* = 4.72$; when $\lambda \to 1$, $\theta^* \to 2.28$; when λ is reduced further, when the asymptotic method [3] ceases to work, we can assume that either $\theta^* \to \infty$ when $2\beta \to \pi$, or the root which leads to this oscillating singularity, on meeting the other root, departs from the pole considered.

This research was supported financially by the Russian Foundation for Basic Research (96-01-00133).

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Translated by R.C.G.